INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 41, Northern Spring 2020 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Is it possible to fill a 40×41 table, with one integer in each 1×1 cell, so that the number in each cell equals the number of cells that have that same number inside and share an edge with that cell? (4 points)

2. Sasha announced after his recent visit to Addis Ababa that he has now celebrated New Year's Eve inside every possible hemisphere of Earth, but one. What is the least number of places where Sasha could have celebrated New Year's Eve? Consider places as points on a sphere. A point on the border of a hemisphere is not counted as inside that hemisphere. (4 points)

3. There are 41 letters around a circle, where each letter is either A or B. It is allowed to replace ABA by B or vice versa, or to replace BAB by A or vice versa. Is it true that from any initial location of the letters one can obtain a single letter on the circle using these operations? (5 points)

4. Does there exist a non-constant polynomial $p(x)$ with real coefficients such that it can be represented as the sum $a(x) + b(x)$, where $a(x)$ and $b(x)$ are the squares of some polynomials with real coefficients,

   (a) in one way only? (2 points)

   (b) in exactly two ways? (3 points)

   Ways that differ only in the order of their terms are considered to be the same.

5. Let two circles intersect at points P and Q. An arbitrarily chosen line ℓ through Q intersects the circles again at points A and B. Let C be a point where the tangents to the circles at points A and B intersect, and let D be a point where the angle bisector of $\angle CPQ$ and line AB meet. Prove that, no matter the choice of ℓ, all possible positions of the point D lie on a single circle. (5 points)